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The principal rank characteristic sequence of a real symmetric matrix

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ABSTRACT

Given a vector $u \in \mathbb{R}^{2^n}$, the principal minor assignment problem asks when is there an $n \times n$ matrix having its 2^n principal minors given by u. This paper explores the following related problem. Given a sequence $r_0 r_1 \cdots r_n$ of 0s and 1s, does there exist an $n \times n$ real symmetric matrix that has a principal submatrix of rank k if and only if $r_k = 1$, for all $0 \leqslant k \leqslant n$? Certain conditions are shown to be necessary in order for this question to have an affirmative answer. Several families of matrices are constructed to attain certain classes of sequences. The problem is solved completely for $n \leq 6$, and for $7 \leq n \leq 10$ in the case of sequences beginning with 010.

1. Introduction

For a fixed value of n and any fixed subset $K \subseteq \{0, \ldots, n\}$, we address the question of determining the existence or nonexistence of an $n \times n$ real symmetric matrix having a principal submatrix of rank k exactly when $k \in K$. To address this question, we first introduce some notation.

Let $A = [a_{ij}]$ be an $n \times n$ matrix and let $X, Y \subseteq \{1, 2, \ldots, n\}$. Then the submatrix of A determined by the rows with indices in X and the columns with indices in Y is denoted by $A[X, Y]$. If $X = Y$, then $A[X, X]$ is a *principal* submatrix of A and we abbreviate this to $A[X]$. We also write $A(X)$ for $A[\bar{X}]$, where $\bar{X} = \{1, 2, \ldots, n\} \setminus X$.

A *minor* of A is the determinant of a square submatrix of A, and the determinant of a principal submatrix is a *principal minor*. The *order* of a minor is k if it is the determinant of a $k \times k$ submatrix. For

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the sake of brevity, we use the notation A_X to denote the principal minor $\det A[X]$. Further, given indices $i_1, i_2, \ldots, i_k$, we write $A_{i_1 i_2 \cdots i_k}$ in place of $A_{\{i_1, i_2, \ldots, i_k\}}$. (Note that the order in which these indices are written makes no difference, since a simultaneous permutation of the rows and columns of a matrix has no effect on its determinant.)

Algebraic relations between the minors of a matrix have been studied for a long time, and there are many well-known relations [4]. If A is nonsingular, a classical identity of Jacobi relates the minors of A to those of A^{-1}. In the case of principal minors, it asserts that for $X \subseteq \{1, 2, \ldots, n\}$,

$$\det A[X] = (\det A)\left(\det A^{-1}(X)\right). \tag{1.1}$$

Muir's law of extensible minors [10] asserts that given a homogeneous polynomial identity in the minors of an $n \times n$ matrix in which an index i does not appear in any minor $\det A[X, Y]$ that occurs in the identity, a new polynomial identity arises by including i in all minors, that is, by replacing each $\det A[X, Y]$ with $\det A[X \cup \{i\}, Y \cup \{i\}]$.

As an illustration of this principle, consider the following identity applied to an $n \times n$ matrix A and two indices $p, q \in \{1, \ldots, n\}$:

$$A_{\emptyset} A_{pq} = A_p A_q - (\det A[\{p\}, \{q\}])(\det A[\{q\}, \{p\}]).$$

(Note that the "empty minor" $A_{\emptyset}$ is defined to have the value 1.) This is simply a restatement of the formula for the determinant of a 2×2 matrix. If $k \le n$ and $\{1, \ldots, k\}$ includes neither p nor q, then by the law of extensible minors,

$$\begin{aligned} A_{1\cdots k} A_{1\cdots kpq} = A_{1\cdots kp} A_{1\cdots kq} &- (\det A[\{1, \ldots, k, p\}, \{1, \ldots, k, q\}]) \\ &\times (\det A[\{1, \ldots, k, q\}, \{1, \ldots, k, p\}]). \end{aligned} \tag{1.2}$$

If A is an $n \times n$ matrix, then A has 2^n principal minors (including $A_{\emptyset} = \det A[\emptyset] = 1$ and $\det A$ itself) and these can be assembled into a real vector in $\mathbb{R}^{2^n}$ (using e.g. lexicographic order) called the *principal minor vector* $\mathrm{pmv}(A)$ of A. The *principal minor assignment problem* introduced in [7] is: Given a vector $u \in \mathbb{R}^{2^n}$, when is there an $n \times n$ matrix A with $\mathrm{pmv}(A) = u$ and, if it exists, how can it be constructed? Several recent papers [5,8,9] have contributed toward the goal of an algebraic characterization of the set of principal minor vectors of the $n \times n$ real matrices.

In this paper, we focus on the case in which A is a real symmetric matrix, and introduce a "characteristic" vector associated to A that throws away most of the information in $\mathrm{pmv}(A)$ and leaves only information about the presence or absence of a nonzero principal minor of each order.

Definition 1.1. The *principal rank characteristic sequence* of an $n \times n$ real symmetric matrix A is defined to be $\mathrm{pr}(A) = r_0 r_1 r_2 \cdots r_n$ where for $0 \le k \le n$,

$$r_k = \begin{cases} 1 & \text{if } A \text{ has a principal submatrix of rank } k, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

We recall that the rank of a matrix A is equal to the maximum k such that A has a nonzero minor of order k, and that the rank of a symmetric matrix A is equal to the maximum k such that A has a nonzero principal minor of order k. This leads to an alternative characterization of the principal rank characteristic sequence of an $n \times n$ real symmetric matrix, namely that for $1 \leqslant k \leqslant n$,

$$r_k = \begin{cases} 1 & \text{if } A \text{ has a nonzero principal minor of order } k, \text{ and} \\ 0 & \text{otherwise,} \end{cases} \tag{1.3}$$

while $r_0 = 1$ if and only if A has a zero on its main diagonal. Thus the principal rank characteristic sequence includes a specification of the zero-nonzero character of the main diagonal in that (i) A has only zero diagonal entries if $r_1 = 0$ (and so $r_0 = 1$), (ii) A has only nonzero diagonal entries if $r_0 = 0$

(and so $r_1 = 1$), and (iii) A has at least one zero and at least one nonzero diagonal entry if $r_0 = 1$ and $r_1 = 1$. Note that it is impossible to have both $r_0 = 0$ and $r_1 = 0$.

The *inverse principal rank characteristic problem* for real symmetric matrices is:

> Given a sequence $\sigma \in \{0, 1\}^{n+1}$, when is there an $n \times n$ real symmetric matrix A with $\mathrm{pr}(A) = \sigma$? That is, determine the set
>
> $$\mathrm{PR}_n = \{\mathrm{pr}(A) : A \text{ an } n \times n \text{ real symmetric matrix}\}$$
>
> and, for each $\sigma \in \mathrm{PR}_n$, construct an $n \times n$ real symmetric matrix A such that $\mathrm{pr}(A) = \sigma$.

We say that a sequence $\sigma \in \{0, 1\}^{n+1}$ is *attainable* provided there exists an $n \times n$ real symmetric matrix A with $\mathrm{pr}(A) = \sigma$. For example, the sequences $100 \cdots 0$ and $011 \cdots 1$ of length $n + 1$ are attainable, as they are the principal rank characteristic sequences of the $n \times n$ zero matrix and identity matrix, respectively. On the other hand, if $r_0 = r_1 = 0$, then $r_0 r_1 r_2 \cdots r_n$ is not attainable, as noted above.

In this paper we give constructions showing various families of sequences to be attainable, and prove several theorems showing that certain other families are not attainable. We determine PR_n explicitly for $n \leqslant 6$, and with the aid of a computer settle the attainability of all sequences of orders $n = 7, 8, 9, 10$ that begin with 010. (These are the sequences belonging to symmetric matrices with only nonzero diagonal entries and all principal minors of order 2 equal to 0.)

2. Basic results and simple constructions

We begin by observing the sequences belonging to some particularly simple $n \times n$ matrices, namely the zero matrix O_n, the identity matrix I_n, and the matrix J_n with every entry equal to 1.

Observation 2.1. *Let $n \geqslant 1$.*

(*i*) $\mathrm{pr}(O_n) = 10 \cdots 0$.
(*ii*) $\mathrm{pr}(I_n) = 01 \cdots 1$.
(*iii*) $\mathrm{pr}(J_n) = 010 \cdots 0$.

Theorem 2.2. *Let $n \geqslant 2$. Let $A = J_n - kI_n$ for some positive integer k, and let $\mathrm{pr}(A) = r_0 r_1 \cdots r_n$. If $k = 1$, then $r_i = 0$ if and only if $i = 1$. If $2 \leq k \leq n$, then $r_i = 0$ if and only if $i = 0$ or $i = k$.*

Proof. For $1 \leqslant \ell \leqslant n$ and $k \geqslant 1$, every $\ell \times \ell$ principal submatrix of A is equal to $J_\ell - kI_\ell$, which has exactly two distinct eigenvalues, namely $\ell - k$ and $-k$. The latter is nonzero, by hypothesis. If $\ell \neq k$, then $J_\ell - kI_\ell$ is nonsingular, and hence $r_\ell = 1$. If $\ell = k$, then this matrix is singular, and hence $r_\ell = 0$. Finally, it is easy to see that $r_0 = 1$ if $k = 1$, whereas $r_0 = 0$ if $k \geqslant 2$. $\square$

If the principal rank characteristic sequence of a matrix is known, then the sequence belonging to the direct sum of the matrix with a zero matrix is easy to determine.

Theorem 2.3. *If $\mathrm{pr}(A) = r_0 r_1 \cdots r_n$ and $k \geqslant 1$, then $\mathrm{pr}(A \oplus O_k) = 1 r_1 r_2 \cdots r_n 0 \cdots 0$.*

Proof. Let $\mathrm{pr}(A \oplus O_k) = r'_0 r'_1 r'_2 \cdots r'_{n+k}$. Obviously, $r'_0 = 1$. For i with $1 \leqslant i \leqslant n$, it is clear that $r_i = 1$ implies $r'_i = 1$. Also, if $r_i = 0$ then every $i \times i$ principal submatrix of A is singular, which implies that every $i \times i$ principal submatrix of $A \oplus O_k$ is singular, and hence $r'_i = 0$. Finally, for $n + 1 \leqslant i \leqslant n + k$, every $i \times i$ principal submatrix has a zero row and column, so $r_i = 0$. $\square$

If instead the direct sum is taken with an identity matrix, then determining the resulting sequence is as simple as determining the sumset of two sets of integers, where the *sumset* of S and T is $\{s + t : s \in S \text{ and } t \in T\}$.

Theorem 2.4. *Suppose* $\mathrm{pr}(A) = r_0r_1 \cdots r_n$ *and* $k \geqslant 1$. *Let* $\mathrm{pr}(A \oplus I_k) = r'_0r'_1r'_2 \cdots r'_{n+k}$. *Then* $r'_0 = r_0$, $r'_1 = \cdots = r'_k = 1$, *and for i with* $k+1 \leqslant i \leqslant n+k$, $r'_i = 1$ *if and only if i is in the sumset of* $\{1 \leqslant j \leqslant n | r_j = 1\}$ *and* $\{0, \ldots, k\}$.

Proof. The first two assertions are clear, so assume $k+1 \leqslant i \leqslant n+k$. Suppose first that $i = t+s$, for $t \in \{1 \leqslant j \leqslant n | r_j = 1\}$ and $s \in \{0, \ldots, k\}$. Then there is some $X \subseteq \{1, \ldots, n\}$ with $|X| = t$ and $A[X]$ nonsingular, so $r'_t = 1$. Thus, for $Y = \{n+j | 1 \leqslant j \leqslant s\}$, the $i \times i$ matrix

$$(A \oplus I_k)[X \cup Y]$$

is clearly nonsingular as well, and so $r'_i = r'_{t+s} = 1$.

Conversely, suppose $r'_i = 1$. Then there exists some $Z \subseteq \{1, \ldots, n+k\}$ with $|Z| = i$ and $(A \oplus I_k)[Z]$ nonsingular. Write $Z = X \cup Y$ where $X \subseteq \{1 \ \ldots, n\}$ and $Y \subseteq \{n+1, \ldots, n+k\}$. Let $Y' = \{j : n+j \in Y\}$. Then

$$(A \oplus I_k)[X \cup Y] = A[X] \oplus I_k[Y']$$

is nonsingular, and hence $A[X]$ is nonsingular. Thus, $t = |X|$ and $s = |Y'| = i - t$ give $i = t + s$, with $t \in \{1 \leqslant j \leqslant n | r_j = 1\}$ and $s \in \{0, \ldots, k\}$. $\square$

We can apply Theorem 2.4 to see that one of the simplest sequences, that in which every term is 1, is attainable.

Corollary 2.5. *Let* $L_2 = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. *Then* $\mathrm{pr}(L_2) = 111$ *and, for* $n \geqslant 3$, $\mathrm{pr}(L_2 \oplus I_{n-2}) = 11 \cdots 1$.

We next show by an appropriate construction that if a given sequence is attainable, appending any number of 0s to the end of the sequence results in another attainable sequence.

Theorem 2.6. *If* $\mathrm{pr}(A) = r_0r_1 \ldots r_n$ *and*

$$A' = \left[\begin{array}{c|c} \begin{matrix} & & \\ & A & \\ & & \end{matrix} & \begin{matrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{matrix} \\ \hline \begin{matrix} a_{1n} & a_{2n} & \cdots & a_{nn} \end{matrix} & a_{nn} \end{array}\right],$$

then $\mathrm{pr}(A') = r_0r_1 \ldots r_n0$. *In particular, appending* 0 *to an attainable sequence results in another attainable sequence.*

Proof. Let $\mathrm{pr}(A') = r'_0r'_1 \cdots r'_{n+1}$. Clearly, $r'_0 = r_0$, $r'_1 = r_1$ and $r'_{n+1} = 0$. Let $2 \leqslant i \leqslant n$. Since the last two rows (and columns) of A' are identical, A' has an $i \times i$ nonsingular principal submatrix if and only if A does. Thus, $r'_i = r_i$. $\square$

Jacobi's identity (1.1) can be used to relate the principal rank characteristic sequence of an invertible matrix to the sequence belonging to the inverse of the matrix.

Theorem 2.7. *Suppose A is an* $n \times n$ *nonsingular real symmetric matrix with* $\mathrm{pr}(A) = r_0r_1 \cdots r_n$. *Let* $\mathrm{pr}(A^{-1}) = r'_0r'_1 \cdots r'_n$. *Then* $r'_n = r_n = 1$, *while for each i with* $1 \leqslant i \leqslant n-1$, $r'_i = r_{n-i}$. *Finally,* $r'_0 = 1$ *if and only if A has some principal minor of order* $n-1$ *that is zero.*

Proof. Clearly $r'_n = r_n = 1$. For i with $1 \leqslant i \leqslant n-1$, it follows from (1.1) that $r'_i = 1$ if and only if $r_{n-i} = 1$. By that identity again, the existence of a zero on the diagonal of A^{-1} is equivalent to the existence of some principal minor of A of order $n-1$ that is zero. $\square$

Theorem 2.7 is useful for showing that certain sequences are attainable; Theorem 2.8 serves as a first illustration of this.

Theorem 2.8. *For $n \geqslant 4$, let*

$$B_n = \left[\begin{array}{ccc|c} & & & 1 \\ & & & 0 \\ & J_{n-1} - I_{n-1} & & \vdots \\ & & & 0 \\ \hline 1 & 0 & \cdots \quad 0 & 0 \end{array}\right].$$

Then $\mathrm{pr}(B_n^{-1}) = 11 \cdots 101$.

Proof. By Theorem 2.2, $\mathrm{pr}(J_{n-1} - I_{n-1}) = 1011 \cdots 1$, hence $\mathrm{pr}(B_n) = 1011 \cdots 1r_n$. Expanding the determinant of B_n along the last row shows that $r_n = 1$. Hence, $\mathrm{pr}(B_n) = 1011 \cdots 1$. Theorem 2.7 then implies that $\mathrm{pr}(B_n^{-1}) = r'_0 1 \cdots 101$, where $r'_0 = 1$ if and only if B_n has some principal minor of order $n-1$ that is zero. But the trailing $(n-1) \times (n-1)$ principal submatrix of B_n is clearly singular, so indeed $r'_0 = 1$. $\square$

3. Graph-theoretic constructions

This section deals with sequences shown to be attainable by consideration of the adjacency matrices belonging to some appropriate families of graphs. Given a graph G, let $A(G)$ denote the adjacency matrix of G. The following observation about the principal rank characteristic sequence of an adjacency matrix is used without remark in the proofs of results of this section.

Observation 3.1. *For any graph G on n vertices, if* $\mathrm{pr}(A(G)) = r_0 r_1 r_2 \cdots r_n$*, then $r_0 = 1$ and $r_1 = 0$.*

If A is an $n \times n$ matrix and $i_1 i_2 \cdots i_n$ is any permutation of $\{1, 2, \ldots, n\}$, then $\{a_{1i_1}, a_{2i_2}, \ldots, a_{ni_n}\}$ is a *transversal* of A. We consider a *transversal of a graph* to be a transversal in its adjacency matrix that does not include any zero entries. It is important to note that such a transversal corresponds to a cover of the vertices of the graph by some vertex-disjoint collection of cycles and edges. (Such a transversal that does not include any cycles thus corresponds to a *perfect matching* in the graph.)

Definition 3.2. A matrix is *combinatorially singular* if every one of its transversals contains a zero entry. A graph G is *combinatorially singular* if its adjacency matrix is combinatorially singular.

Let K_n denote the complete graph on n vertices. Then $A(K_n) = J_n - I_n$. Hence $\mathrm{pr}(A(K_n)) = 1011 \cdots 1$ by Theorem 2.2. Let P_n and C_n denote the path and cycle, respectively, on n vertices. The next two results give the principal rank characteristic sequences associated with the adjacency matrices of these graphs.

Lemma 3.3. *For $n \geqslant 2$, let* $\mathrm{pr}(A(P_n)) = r_0 r_1 r_2 \cdots r_n$*. Then $r_i = 1$ if and only if i is even.*

Proof. For $n \geqslant 2$, let $A = A(P_n)$. If k is odd, then any $k \times k$ principal submatrix B of A is the adjacency matrix of a subgraph of P_n on an odd number of vertices. Such a subgraph cannot have a perfect matching, but a graph containing neither a cycle nor a perfect matching must be combinatorially singular. Hence, B is singular. As B was chosen arbitrarily, this shows that $r_k = 0$.

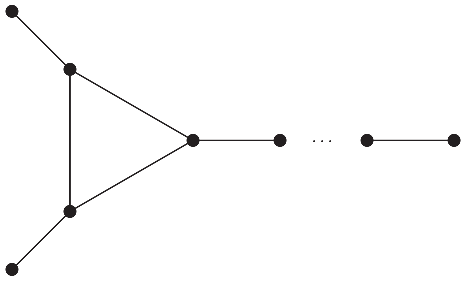

Fig. 3.1. The graph V_n.

If $k \geqslant 2$ is even, then P_n contains P_k as a subgraph, so $B = A(P_k)$ is a submatrix of A. Since P_k is a path on an even number of vertices, it has a unique perfect matching. As P_k is acyclic, this implies that the determinant of B has exactly one nonzero term, and thus is nonzero. Hence, $r_k = 1$. $\square$

Lemma 3.4. *For $n \geqslant 3$, let $\mathrm{pr}(A(C_n)) = r_0r_1r_2 \cdots r_n$. Then $r_n = 0$ if and only if n is a multiple of 4 while, for $0 \leqslant i \leqslant n-1$, $r_i = 1$ if and only if i is even.*

Proof. Let $A = A(C_n)$. Suppose first that $1 \leqslant k \leqslant n-1$. If k is even, note that C_n has P_k as a subgraph, and hence by Lemma 3.3, the principal submatrix corresponding to this subgraph is nonsingular. Thus, $r_k = 1$. If k is odd, then let B be any $k \times k$ principal submatrix of A. Then B corresponds to some subgraph of C_n with k vertices. But this subgraph can be thought of as a subgraph of P_{n-1}, and so Lemma 3.3 shows that B must be singular, and thus $r_k = 0$.

The eigenvalues of A are $2\cos(2\pi j/n)$ for $j = 1, \ldots, n$ (see, for example, [6, Example 7, p. 28-7]) If n is a multiple of 4, then $j = n/4$ gives a zero eigenvalue, hence $r_n = 0$. If n is not a multiple of 4, then A is nonsingular, giving $r_n = 1$. $\square$

In Lemmas 3.5 and 3.6, graphs are constructed to yield specific sequences of interest. The sequences accounted for by Lemma 3.6 are of particular relevance to Question 6.6.

Lemma 3.5. *For every even n with $n \geqslant 4$, let F_n be the graph on n vertices formed by adding a pendent edge to C_{n-1}. Let $A = A(F_n)$. Then $\mathrm{pr}(A) = 1010 \cdots 10111$ and $\mathrm{pr}(A^{-1}) = 1110101 \cdots 01$.*

Proof. Let $\mathrm{pr}(A) = r_0r_1r_2 \cdots r_n$. Consider $1 \leqslant k \leqslant n-1$. Since A has a principal submatrix equal to $A(C_{n-1})$, by Lemma 3.4, $r_{n-1} = 1$ and $r_k = 1$ if k is even. For k odd and $k \leqslant n-3$, the graph of every $k \times k$ principal submatrix of A is a forest on an odd number of vertices, and hence must be combinatorially singular. Thus, $r_k = 0$ for k odd. Finally, $r_n = 1$, as the graph F_n has a unique perfect matching and hence there is precisely one nonzero term in the determinant of A.

As the subgraph of F_n that is obtained by deleting the neighbor of its pendent vertex is combinatorially singular, A has a principal minor of order $n-1$ that is zero. It now follows by Theorem 2.7 that $\mathrm{pr}(A^{-1}) = 1110101 \cdots 01$. $\square$

Lemma 3.6. *For every even n with $n \geqslant 6$, let V_n be the graph constructed by starting with the cycle C_3, adding a pendent edge at each of two of its vertices, and adding a pendent path of length $n-5$ at the remaining vertex. (See Figure 3.1.) Then $\mathrm{pr}(A(V_n)) = 1011 \cdots 101$.*

Proof. Let $\mathrm{pr}(A(V_n)) = r_0r_1 \cdots r_n$. To see that $r_n = 1$, note that V_n has a unique transversal, since any transversal must contain the two pendent edges connected to the cycle, and the remaining $n-4$ vertices induce a path that itself has a unique transversal. Hence, there is precisely one nonzero term in the determinant of $A(V_n)$, giving $r_n = 1$.

To see that $r_{n-1} = 0$, consider a subgraph of V_n formed by deleting a single vertex. If the deleted vertex is part of the cycle, then the remaining graph is a forest on an odd number of vertices, and hence is combinatorially singular. Otherwise, the cycle must be included in any transversal. But a transversal including the cycle must leave the original pendent vertices unmatched. Hence, the subgraph must be combinatorially singular, and so $r_{n-1} = 0$.

Finally, suppose $2 \leqslant k \leqslant n-2$. If k is odd, then in fact $k \leqslant n-3$. Consider the subgraph consisting of the cycle and a path on $k-3$ other vertices. By Lemmas 3.3 and 3.4, the principal submatrix corresponding to this subgraph is nonsingular, and so $r_k = 1$. On the other hand, if k is even, then P_k is a subgraph of V_n and hence, by Lemma 3.3, $r_k = 1$ in this case as well. $\square$

The purpose of the next construction is to exhibit attainable sequences with a single 0 entry in a specific position.

Theorem 3.7. *For each $m \geqslant 1$, define the graph G_m to be a perfect matching on m vertices when m is even, and the disjoint union of G_{m-1} and an isolated vertex when m is odd. For n and k positive integers with $n \geqslant 3$, let*

$$Q_{n,k} = \left[\begin{array}{c|ccc} 2k & 1 & \cdots & 1 \\ \hline 1 & & & \\ \vdots & & A(G_{n-1}) & \\ 1 & & & \end{array}\right].$$

If $\mathrm{pr}(Q_{n,k}) = r_0 r_1 \cdots r_n$, then $r_i = 0$ if and only if $i = 2k+1$.

Proof. Let $\mathrm{pr}(Q_{n,k}) = r_0 r_1 \cdots r_n$. Clearly $r_0 = r_1 = 1$, so suppose $2 \leqslant i \leqslant n$. Note that for m even, $A(G_m)$ is nonsingular with determinant $(-1)^{m/2}$.

First consider the case of i even. In this case, certainly $i \neq 2k+1$, so we need to show $r_i = 1$. If $i \leqslant n-1$, then $Q_{n,k}$ has a principal submatrix equal to $A(G_i)$, and so $r_i = 1$. On the other hand, if $i = n$, then $r_i = 1$ if $\det Q_{n,k} \neq 0$. To evaluate this determinant, consider the result of subtracting each of rows 2 through n from the first row, then subtracting each of columns 2 through n from the first column. This produces the matrix

$$\left[\begin{array}{c|ccc|c} 2k-n & 0 & \cdots & 0 & 1 \\ \hline 0 & & & & 0 \\ \vdots & & A(G_{n-2}) & & \vdots \\ 0 & & & & 0 \\ \hline 1 & 0 & \cdots & 0 & 0 \end{array}\right],$$

and the determinant of this matrix, and hence of $Q_{n,k}$, is $-\det A(G_{n-2})$. As $n = i$ is even, this is nonzero, and hence $r_i = 1$ in this case as well.

Now suppose i is odd. If $r_i = 0$ then, in particular, the leading principal $i \times i$ submatrix of $Q_{n,k}$ has a zero determinant. This submatrix is $Q_{i,k}$. To evaluate the determinant of this matrix, consider the result of subtracting each of rows 2 through i from the first row, then subtracting each of columns 2 through i from the first column. This yields

$$0 = \det(Q_{i,k}) = \det \left[\begin{array}{c|ccc} 2k-(i-1) & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & A(G_{i-1}) & \\ 0 & & & \end{array}\right] = (-1)^{\frac{i-1}{2}}(2k-(i-1)), \tag{3.1}$$

and so $i = 2k + 1$. On the other hand, if $r_i = 1$, then there is some $X \subseteq \{1, \ldots, n\}$ with $|X| = i$ such that $Q_{n,k}[X]$ is nonsingular. As i is odd, G_{n-1} does not contain any transversal on i vertices, thus $A(G_{n-1})$ has no $i \times i$ nonsingular principal submatrix. Hence $1 \in X$. Let $X' = X \setminus \{1\}$ and consider the subgraph G' of G_{n-1} induced by those vertices corresponding to the indices in X'. If this subgraph includes two isolated vertices, then $Q_{n,k}[X']$ has two zero rows, implying that $Q_{n,k}[X]$ is singular, a contradiction. Hence, G' has no isolated vertices, and is therefore a matching. Thus, $Q_{n,k}[X] = Q_{i,k}$. As the determinant of this submatrix is nonsingular, it follows from (3.1) that $i \neq 2k + 1$. □

4. Forbidden subsequences

The purpose of this section is to highlight certain subsequences which, if present in a sequence, indicate that the sequence is not attainable. Our first result in this direction shows that a sequence of the form $1 * 0 * \cdots * 1$ (where $*$ indicates a single unspecified term) is never attainable.

Theorem 4.1. *Let* $\mathrm{pr}(A) = r_0 r_1 \cdots r_n$, *and suppose* $r_0 = 1$ *while* $r_2 = 0$. *Then A is singular, i.e.* $r_n = 0$.

Proof. Since $r_0 = 1$, A must have some zero entry on its diagonal, say $a_{ii} = 0$. Now consider any $j \neq i$ with $1 \leqslant j \leqslant n$. Since $r_2 = 0$, the minor A_{ij} is zero. But since $a_{ii} = 0$, this requires that $a_{ij} = a_{ji} = 0$. By the choice of j, this shows that the ith row and the ith column of A contain only zero entries. This certainly implies that A is singular, and hence $r_n = 0$. □

Theorem 4.3 below appears in [3]. Since this result does not seem to be well known, we include a proof, using the following lemma.

Lemma 4.2. *Let A be an* $n \times n$ *real symmetric matrix such that for some r with* $1 \leqslant r \leqslant n - 1$, *the principal minor* $A_{1 \ldots r}$ *is nonzero. Suppose that for all* $p, q \in \{r + 1, \ldots, n\}$,

$$\det A[\{1, \ldots, r, p\}, \{1, \ldots, r, q\}] = 0.$$

Then the rank of A is r, and thus every minor of A of order $r + 1$ *is zero.*

Proof. Let p be an integer with $r + 1 \leqslant p \leqslant n$. By hypothesis, the first r columns of the submatrix $A[\{1, \ldots, r, p\}, \{1, \ldots, n\}]$ are linearly independent. Therefore, if this submatrix had rank $r + 1$, then it would have some qth column not in the span of the first r columns, and then $A[\{1, \ldots, r, p\}, \{1, \ldots, r, q\}]$ would be nonsingular, contradicting the hypothesis. Hence, the submatrix $A[\{1, \ldots, r, p\}, \{1, \ldots, n\}]$ must have rank r. As the first r rows of this submatrix are linearly independent, row p is in the span of the first r rows. By the choice of p, it follows that all of rows $r + 1$ through n of A are in the span of the first r rows. Thus, A has rank r, and so every $(r + 1) \times (r + 1)$ submatrix of A is singular. □

Theorem 4.3 (Bôcher [3, Section 20 Theorem 1])**.** *Let A be an* $n \times n$ *real symmetric matrix. Let* A' *be some* $k \times k$ *nonsingular principal submatrix of A. If* A' *is contained in any larger nonsingular principal submatrix of A, then it must be contained in one of dimension* $(k+1) \times (k+1)$ *or one of dimension* $(k+2) \times (k+2)$. *Otherwise, the rank of A is k.*

Proof. Assume without loss of generality that $A' = A[\{1, \ldots, k\}]$ and suppose every principal submatrix of A of dimension $(k + 1) \times (k + 1)$ or $(k + 2) \times (k + 2)$ containing A' is singular. It then follows from (1.2) that

$$\det A[\{1, \ldots, k, p\}, \{1, \ldots, k, q\}] = 0$$

for any $p, q \in \{k + 1, \ldots, n\}$, and so by Lemma 4.2 the rank of A is k. □

Theorem 4.3 is of interest here because it can be used to show that the sequence 001 cannot occur as a subsequence of any attainable sequence; this is the content of the next result.

Theorem 4.4. *Let* $\mathrm{pr}(A) = r_0 r_1 \cdots r_n$ *and suppose that, for some* k *with* $0 \leqslant k \leqslant n-2$, $r_{k+1} = r_{k+2} = 0$. *Then* $r_i = 0$ *for all* $i \geqslant k+1$. *In particular,* $r_n = 0$, *so that* A *is singular.*

Proof. Let k be minimal such that $r_{k+1} = r_{k+2} = 0$. Then $r_k = 1$, so assume with no loss of generality that the leading $k \times k$ principal submatrix of A is nonsingular. It then follows by Theorem 4.3 that the rank of A is k. Hence, every square submatrix of A of dimension exceeding k is singular, and so $r_{k+1} = r_{k+2} = \cdots = r_n = 0$. $\square$

Suppose A is an $n \times n$ real symmetric matrix with $\mathrm{pr}(A) = r_0 r_1 \cdots r_n$ and let A' be a $k \times k$ principal submatrix of A with $\mathrm{pr}(A') = r'_0 r'_1 \cdots r'_k$. If $r_j = 0$ for some $j \leqslant k$, then certainly $r'_j = 0$, because if A does not have a nonzero principal minor of order j, then neither does A'. On the other hand, if $r_j = 1$, it is certainly possible that $r'_j = 0$. One powerful application of Theorem 4.4, however, is to limit when this may happen; if $r'_j = 0$ then it cannot be the case that $r'_{j-1} = 0$ or $r'_{j+1} = 0$ as well unless $r'_j = r'_{j+1} = \cdots = r_k = 0$. But this must happen if $r_{j-1} r_j r_{j+1} \neq 111$. Hence, if we restrict the occurrence of three consecutive 1s in $\mathrm{pr}(A)$, we can ensure that certain 1s in this sequence are preserved by the sequence of A'. This is the essence of the following lemma, and it is an idea that appears again in the proof of Lemma 7.1.

Lemma 4.5. *Let* $\mathrm{pr}(A) = r_0 r_1 \cdots r_n$ *and suppose that, for some* l *and* m *with* $0 \leqslant l < m \leqslant n$, *the sequence* $r_l r_{l+1} \cdots r_m$ *does not contain three consecutive 1s. Then there exists some* $m \times m$ *principal submatrix* A' *of* A *such that if* $\mathrm{pr}(A') = r'_0 r'_1 \cdots r'_m$, *then* $r'_{l+1} r'_{l+2} \cdots r'_m = r_{l+1} r_{l+2} \cdots r_m$.

Proof. Let $k \leqslant m$ be largest such that $r_k = 1$, and let B be any nonsingular $k \times k$ principal submatrix of A. Let $\mathrm{pr}(B) = s_0 s_1 \cdots s_k$. We claim that $s_i = r_i$ for each i with $l+1 \leqslant i \leqslant k$. If $r_i = 0$, then this is clear. So assume $r_i = 1$. If $i = k$ then $s_i = 1 = r_i$ because B is nonsingular. Otherwise, $l+1 \leqslant i \leqslant k-1$ and then, as $r_l r_{l+1} \cdots r_k$ does not contain three consecutive 1s, either $r_{i-1} = 0$ or $r_{i+1} = 0$. Hence, either $s_{i-1} = 0$ or $s_{i+1} = 0$. If $s_{i+1} = 0$, then $s_i = 1$, as otherwise B would violate Theorem 4.4. If $s_{i-1} = 0$, then in the case $i \geqslant 2$, again $s_i = 1$ to avoid contradicting Theorem 4.4. On the other hand, if $i = 1$, then in fact $s_0 = 0$, implying that $s_1 = 1$.

Now take A' to be some $m \times m$ principal submatrix of A containing B. Then with $\mathrm{pr}(A') = r'_0 r'_1 \cdots r'_m$,

$$r'_{l+1} r'_{l+2} \cdots r'_k = s_{l+1} s_{l+2} \cdots s_k = r_{l+1} r_{l+2} \cdots r_k,$$

while the choice of k implies that $r_i = 0$ and hence $r'_i = 0$ for i with $k+1 \leqslant i \leqslant m$. Thus, $r_{k+1} r_{k+2} \cdots r_m = r'_{k+1} r'_{k+2} \cdots r'_m$ as well, and the proof is complete. $\square$

Given a sequence that is not attainable, the following theorem gives a set of additional hypotheses which are enough to imply that in fact the sequence does not even occur as a subsequence of some longer attainable sequence. Our primary motivation for including this result is its application in Section 6.

Theorem 4.6. *Let* σ *be a sequence that is not attainable. If additionally*

(*i*) σ *does not contain three consecutive 1s,*
(*ii*) σ *does not have* 11 *as its initial subsequence,*
(*iii*) σ *has* 01 *as its terminal subsequence, and*
(*iv*) *the reverse sequence of* σ *is not attainable,*

then σ *does not occur as a subsequence of any attainable sequence.*

Proof. Suppose to the contrary that there is some $n \times n$ matrix A such that $\mathrm{pr}(A)$ contains σ as a subsequence. In particular, say $\mathrm{pr}(A) = r_0 r_1 \cdots r_n$ and $\sigma = r_l \cdots r_m$ for some l and m with $0 \leqslant l < m \leqslant n$.

By (i) this subsequence does not contain three consecutive 1s, and so by Lemma 4.5 there is some principal submatrix A' of A with $\mathrm{pr}(A') = r'_0 r'_1 r'_2 \cdots r'_l r'_{l+1} \cdots r'_m$ such that $r'_{l+1} r'_{l+2} \cdots r'_m = r_{l+1} r_{l+2} \cdots r_m$. Note that, as σ ends with 01, it follows that A' is nonsingular.

If σ begins with a 0, then $r_l = 0$ and then clearly $r'_l = 0 = r_l$. Now suppose σ begins with a 1, i.e. that $r_l = 1$. Note that σ does not begin with 110 or 111 by (ii), while σ may not begin with 100, as otherwise A' would violate Theorem 4.1. Hence, σ begins with 101, and in particular $r'_{l+1} = r_{l+1} = 0$. In the case $l \geqslant 1$, by Theorem 4.4 this implies $r'_l = 1 = r_l$. Otherwise, if $l = 0$, then $r'_l = 1 = r_l$ because $\mathrm{pr}(A')$ cannot begin with 00. As $r'_l = r_l$ in every case, this establishes that σ is the terminal subsequence of A'.

Now let $B = (A')^{-1}$ and note that by Theorem 2.7, if $\mathrm{pr}(B) = s_0 s_1 \cdots s_m$, then

$$s_1 s_2 \cdots s_{m-l} = r_{m-1} \cdots r_l.$$

Further, $r_{m-1} = 0$ by (iii), and so $s_1 = 0$, implying that $s_0 = 1 = r_m$. Hence, the initial subsequence $s_0 s_1 \cdots s_{m-l}$ of $\mathrm{pr}(B)$ is the reverse of σ. As this does not contain three consecutive 1s, by Lemma 4.5 there is some $(m-l) \times (m-l)$ submatrix B' of B such that with $\mathrm{pr}(B') = s'_0 s'_1 \cdots s'_{m-l}$,

$$s'_1 s'_2 \cdots s'_{m-l} = s_1 s_2 \cdots s_{m-l} = r_{m-1} \cdots r_l.$$

Again, $s'_1 = s_1 = 0$, and so $s'_0 = 1 = r_m$. Thus, $\mathrm{pr}(B')$ is the reverse of σ, contradicting (iv). $\square$

5. Attainable sequences for $n = 1, 2, 3, 4$

The results developed in the previous sections allow each sequence with $1 \leqslant n \leqslant 4$ to be classified as attainable or not attainable, and provide a matrix realizing each such attainable sequence. Thus, PR_n is determined for $n \leq 4$, and the results are shown in Tables 5.1–5.4 below.

Table 5.1
All attainable sequences for $n = 1$.

Sequence	Matrix	Result(s)
01	I_1	2.1(ii)
10	O_1	2.1(i)

Table 5.2
All attainable sequences for $n = 2$.

Sequence	Matrix	Result(s)
$r_0 r_1 0$ with $r_0 r_1$ attainable	–	2.6
011	I_2	2.1(ii)
101	$A(P_2)$	3.3
110	$J_1 \oplus O_1$	2.1(iii) and 2.3
111	L_2	2.5

Table 5.3
All attainable sequences for $n = 3$.

Sequence	Matrix	Result(s)
$r_0 r_1 r_2 0$ with $r_0 r_1 r_2$ attainable	–	2.6
0101	$J_3 - 2I_3$	2.2
0111	I_3	2.1(ii)
1011	$J_3 - I_3 = A(K_3)$	2.2
1111	$L_2 \oplus I_1$	2.5

Table 5.4
All attainable sequences for $n = 4$.

Sequence	Matrix	Result(s)
$r_0r_1r_2r_3 0$ with $r_0r_1r_2r_3$ attainable	–	2.6
01011	$J_4 - 2I_4$	2.2
01101	$J_4 - 3I_4$	2.2
01111	I_4	2.1(ii)
10101	$A(P_4)$	3.3
10111	$J_4 - I_4 = A(K_4)$	2.2
11010	$(J_3 - 2I_3) \oplus O_1$	2.2 and 2.3
11101	$Q_{4,1}$ or $(B_4)^{-1}$	3.7 or 2.8
11111	$L_2 \oplus I_2$	2.5

6. Sequences of the form 1011 · · · 101

In the previous section, we noted that the results developed here so far were enough to decide the attainability of every sequence for $n \leq 4$. In fact, for $n = 5$ there is only one sequence left unaccounted for. That sequence, namely 101101, requires considerably more work to settle, and leads to an interesting general result. We begin with the following lemma.

Lemma 6.1. *Suppose a* 4×4 *real symmetric matrix A has only zero entries on its diagonal, only nonzero entries elsewhere, and suppose every minor of A of order* 2 *that does not intersect the diagonal is zero. Then A is nonsingular.*

Proof. Suppose

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{12} & 0 & a_{23} & a_{24} \\ a_{13} & a_{23} & 0 & a_{34} \\ a_{14} & a_{24} & a_{34} & 0 \end{bmatrix}.$$

As every minor of A with order 2 that does not intersect the diagonal is equal to zero,

$$a_{13}a_{24} = a_{14}a_{23} = a_{12}a_{34} = a_{13}a_{24}.$$

Call this common value α, note that $\alpha \neq 0$ by hypothesis, and observe that

$$\begin{aligned}\det A &= (a_{12}a_{34})^2 - 2(a_{12}a_{34})(a_{13}a_{24}) - 2(a_{12}a_{34})(a_{14}a_{23}) + (a_{13}a_{24})^2 \\ &\quad - 2(a_{13}a_{24})(a_{14}a_{23}) + (a_{14}a_{23})^2 \\ &= (\alpha)^2 - 2(\alpha)(\alpha) - 2(\alpha)(\alpha) + (\alpha)^2 - 2(\alpha)(\alpha) + (\alpha)^2 \\ &= -3\alpha^2 < 0. \quad \square\end{aligned}$$

Lemma 6.2. *Suppose* $\mathrm{pr}(A) = 101101$. *Let* $X = \{1, \ldots, 5\}$, $I = \{i_1, i_2\} \subset X$ *and* $J = \{j_1, j_2, j_3\} = X \setminus I$. *If* $A_I \neq 0$, *then*

$$-\frac{1}{4}\det A = \frac{A_{I\cup\{j_1\}}A_{I\cup\{j_2\}}A_{I\cup\{j_3\}}}{(A_I)^2} = \frac{8a_{i_1j_1}a_{i_2j_1}a_{i_1j_2}a_{i_2j_2}a_{i_1j_3}a_{i_2j_3}}{a_{i_1i_2}}. \tag{6.1}$$

Proof. It is straightforward to verify that for any symmetric square matrix A of dimension at least 3,

$$\begin{aligned}&(A_{123} - A_{12}A_3 - A_{13}A_2 - A_{23}A_1 + 2A_1A_2A_3)^2 \\ &\quad = 4\,(A_1A_2 - A_{12})(A_2A_3 - A_{23})(A_1A_3 - A_{13}).\end{aligned} \tag{6.2}$$

After expansion and homogenization, this becomes

$$\begin{aligned}0 = &(A_\emptyset)^2(A_{123})^2 + (A_1)^2(A_{23})^2 + (A_2)^2(A_{13})^2 + (A_3)^2(A_{12})^2 + 4\,A_\emptyset A_{12}A_{13}A_{23}\\ &+ 4\,A_1A_2A_3A_{123} - 2A_\emptyset A_1A_{23}A_{123} - 2A_\emptyset A_2A_{13}A_{123} - 2A_\emptyset A_3A_{12}A_{123}\\ &- 2A_1A_2A_{13}A_{23} - 2A_1A_3A_{12}A_{23} - 2A_2A_3A_{12}A_{13}.\end{aligned} \tag{6.3}$$

This identity appears in [8, Eq. (2)] where it is identified as a *hyperdeterminantal relation* of format $2 \times 2 \times 2$. Although the validity of this identity relies upon the symmetry of the matrix A, it is shown in [8, Theorem 2] that expanding this identity as in Muir's law of extensible minors (see Section 1) results in an identity that is again valid for symmetric matrices. Thus, suppose such an expansion is performed by adding the indices 4 and 5 to every minor, and observe that in all but two of the terms of the resulting identity at least one principal minor of order 4 appears as a factor. Therefore, if $\mathrm{pr}(A) = 101101$, all but those two terms vanish, and the resulting identity is

$$(A_{45})^2(A_{12345})^2 + 4\,A_{145}A_{245}A_{345}A_{12345} = 0,$$

or equivalently,

$$(A_{45})^2(\det A) = -4\,A_{145}A_{245}A_{345}.$$

Beginning instead with the identity corresponding to (6.2) but on row and column indices j_1, j_2 and j_3 in place of 1, 2 and 3, and then extending by the indices in I, the identity that results is

$$(A_I)^2(\det A) = -4\,A_{I\cup\{j_1\}}A_{I\cup\{j_2\}}A_{I\cup\{j_3\}}.$$

Solving appropriately gives the first equality in (6.1).

As A has only zero entries on its diagonal, $A_I = -a_{i_1i_2}a_{i_2i_1} = -(a_{i_1i_2})^2$, while each principal minor of A of order 3 is twice the product of its entries above the diagonal. From this, and exploiting the symmetry of A, it follows that

$$\begin{aligned}\frac{A_{I\cup\{j_1\}}A_{I\cup\{j_2\}}A_{I\cup\{j_3\}}}{(A_I)^2} &= \frac{(2a_{i_1i_2}a_{i_2j_1}a_{j_1i_1})(2a_{i_1i_2}a_{i_2j_2}a_{j_2i_1})(2a_{i_1i_2}a_{i_2j_3}a_{j_3i_1})}{(a_{i_1i_2})^4}\\ &= \frac{8a_{i_1j_1}a_{i_2j_1}a_{i_1j_2}a_{i_2j_2}a_{i_1j_3}a_{i_2j_3}}{a_{i_1i_2}}. \quad \square\end{aligned}$$

The next theorem completes the determination of PR_5 by showing that the sequence 101101 is not attainable. This is essentially equivalent to a theorem of Parker [11, Theorem 2] that generalized an earlier result of Blumenthal [1,2]. Parker's original proof was algebraic, and proceeded by manipulating explicit expressions for the principal minors of order 4 of a general 5×5 symmetric matrix with zero diagonal. Later, in [12], Parker gave an entirely different proof using the geometry of quadratic forms.

Theorem 6.3. *There does not exist a real symmetric matrix A such that* $\mathrm{pr}(A) = 101101$.

Proof. Suppose to the contrary that there is some real symmetric matrix A with $\mathrm{pr}(A) = 101101$. Then A has the form

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{12} & 0 & a_{23} & a_{24} & a_{25} \\ a_{13} & a_{23} & 0 & a_{34} & a_{35} \\ a_{14} & a_{24} & a_{34} & 0 & a_{45} \\ a_{15} & a_{25} & a_{35} & a_{45} & 0 \end{bmatrix}.$$

Table 6.1
All attainable sequences for $n = 5$.

Sequence	Matrix	Result(s)
$r_0r_1r_2r_3r_40$ with $r_0r_1r_2r_3r_4$ attainable	-	2.6
010101	$(A(C_5))^{-1}$	3.4 and 2.7
010111	$J_5 - 2I_5$	2.2
011011	$J_5 - 3I_5$	2.2
011101	$J_5 - 4I_5$	2.2
011111	I_5	2.1(ii)
101011	$A(C_5)$	3.4
101111	$J_5 - I_5 = A(K_5)$	2.2
110110	$(J_4 - 2I_4) \oplus O_1$	2.2 and 2.3
111011	$Q_{5,1}$	3.7
111101	$(B_5)^{-1}$	2.8
111111	$L_2 \oplus I_3$	2.5

As $\det A \neq 0$, some off-diagonal entry of A must be nonzero; assume $a_{12} \neq 0$. It then follows by taking $I = \{1, 2\}$ in Lemma 6.2 that every off-diagonal entry in rows and columns 1 and 2 of A must be nonzero. Then, for $k \in \{3, 4\}$, taking $I = \{1, k\}$ in Lemma 6.2 shows that every off-diagonal entry of A in row or column k must be nonzero. Hence, every entry of A off the diagonal is nonzero.

Let $\{i, j, k, l, m\} = \{1, \ldots, 5\}$. Applying Lemma 6.2 with each of $I_1 = \{i, k\}, I_2 = \{i, l\}, I_3 = \{j, k\}$, and $I_4 = \{j, l\}$ in turn yields

$$-\frac{1}{4}\det A = \frac{A_{imk}A_{ijk}A_{ikl}}{(A_{ik})^2} = \frac{A_{iml}A_{ijl}A_{ikl}}{(A_{il})^2} = \frac{A_{ijk}A_{mjk}A_{jkl}}{(A_{jk})^2} = \frac{A_{ijl}A_{mjl}A_{jkl}}{(A_{jl})^2},$$

whence

$$\frac{A_{imk}A_{ijk}A_{ikl}}{(A_{ik})^2} \cdot \frac{A_{ijl}A_{mjl}A_{jkl}}{(A_{jl})^2} = \frac{A_{iml}A_{ijl}A_{ikl}}{(A_{il})^2} \cdot \frac{A_{ijk}A_{mjk}A_{jkl}}{(A_{jk})^2}.$$

After cancellation this yields

$$A_{imk}A_{mjl}(A_{il})^2(A_{jk})^2 = A_{iml}A_{mjk}(A_{ik})^2(A_{jl})^2.$$

Expressing this in terms of the entries of A gives

$$a_{im}a_{mk}a_{ki}a_{mj}a_{jl}a_{lm}a_{il}^4a_{jk}^4 = a_{im}a_{ml}a_{li}a_{mj}a_{jk}a_{km}a_{ik}^4a_{jl}^4.$$

After cancellation (exploiting symmetry) this becomes $(a_{il}a_{jk})^3 = (a_{ik}a_{jl})^3$. As all entries of A are real, this implies that $a_{il}a_{jk} = a_{ik}a_{jl}$. In particular, $\det A[\{i, j\}, \{k, l\}] = 0$. As i, j, k and l were chosen arbitrarily, it follows that every minor of A of order 2 that does not intersect the diagonal is zero. But by Lemma 6.1, this contradicts the hypothesis that A has no nonzero principal minor of order 4. □

With the help of Theorem 6.3, PR_5 is completely determined for $n = 5$; see Table 6.1.

By observing that the sequence 101101 satisfies the hypotheses of Theorem 4.6, we immediately derive the following.

Theorem 6.4. *The sequence* 101101 *does not occur as a subsequence of any attainable sequence.*

The next result, which can be viewed as a corollary of Theorem 6.4, is equivalent to a theorem of Parker in [13, Theorem 3].

Theorem 6.5. *Suppose $n \geqslant 4$ and $\mathrm{pr}(A) = r_0r_1 \cdots r_n$. If, for some k with $1 \leqslant k \leqslant n-3$, $r_k = r_{k+3} = 0$, then $r_i = 0$ for all $k + 3 \leqslant i \leqslant n$. In particular, A is singular.*

Proof. Suppose that $1 \leqslant k \leqslant n - 3$ and $r_k = r_{k+3} = 0$. If $k + 3 = n$ then the conclusion follows trivially, so assume $k \leq n - 4$. If any one of $r_{k-1}, r_{k+1}, r_{k+2}$ or r_{k+4} is 0, then $r_{k-1}r_k \cdots r_{k+4}$

contains two consecutive 0s, in which case it follows by Theorem 4.4 that $r_{k+3} = \cdots = r_n = 0$, and the proof is complete. But the only alternative is $r_{k-1}r_k \cdots r_{k+4} = 101101$, which would contradict Theorem 6.4. $\square$

There is a commonality between Theorems 4.4 and 6.5 that leads to the following question.

Question 6.6. Fix some $s \geq 1$. Is it the case that for any $n \times n$ real symmetric matrix A with $\text{pr}(A) = r_0r_1 \cdots r_n$, if $r_k = r_{k+s} = 0$, then $r_i = 0$ for all i with $k + s \leqslant i \leqslant n$?

Lemma 3.3 for $n = 4$ and Lemma 3.6 for even $n \geqslant 6$ show that Question 6.6 has a negative answer for all even values of s. On the other hand, Theorem 4.4 shows that the answer is affirmative for $s = 1$, while Theorem 6.5 shows that this is also the case for $s = 3$. The next example shows, however, that the answer is negative for $s = 5$.

Example 6.7. Consider the 7×7 circulant matrix

$$A = \begin{bmatrix} 0 & 1 & x & 0 & 0 & x & 1 \\ 1 & 0 & 1 & x & 0 & 0 & x \\ x & 1 & 0 & 1 & x & 0 & 0 \\ 0 & x & 1 & 0 & 1 & x & 0 \\ 0 & 0 & x & 1 & 0 & 1 & x \\ x & 0 & 0 & x & 1 & 0 & 1 \\ 1 & x & 0 & 0 & x & 1 & 0 \end{bmatrix}.$$

It is easy to check that every principal minor of A of order 6 is equal to

$$-x^6 + 4x^5 - 2x^4 - 6x^3 + 7x^2 - 2x - 1 = -(x^3 - 4x^2 + 3x + 1)(x^3 - x + 1), \tag{6.4}$$

while

$$\begin{aligned} \det(A) &= 2x^7 - 14x^6 + 28x^5 - 42x^3 + 14x^2 + 14x + 2 \\ &= (2x^4 - 14x^3 + 30x^2 - 16x + 2)(x^3 - x + 1) - 32x(x - 1). \end{aligned} \tag{6.5}$$

Observe that $x^3 - x + 1$ has a real root that is negative and not 0 or 1. This root provides a value for x that makes (6.4) zero while (6.5) is not zero. Hence, with this value for x, $\text{pr}(A) = 1011r_4r_501$, with $r_5 = 1$ by Theorem 4.4 and $r_4 = 1$ by Theorem 6.5, so that $\text{pr}(A) = 10111101$.

While there is no barrier to extending the definition of $\text{pr}(A)$ to complex Hermitian matrices in the natural way, Theorem 6.3 fails for this extended definition. (And thus, Theorems 6.4 and 6.5 fail as well.) To demonstrate this, we draw on the work [15] of Thompson, who performs a simple but clever calculation using the compound matrix to derive an equation that relates the characteristic polynomial of a principal submatrix of a diagonalizable matrix A to the characteristic polynomial of A itself. In the special case where A is normal with eigenvalues $\lambda_1, \ldots, \lambda_n$, this equation [15, Eq. (8)] gives

$$f_{(i)}(x) = \sum_{j=1}^{n} |U_{ij}|^2 f(x)(x - \lambda_j)^{-1}, \tag{6.6}$$

while [15, Eq. (2)] gives

$$f_{[i]}(x) = \sum_{j=1}^{n} |U_{ij}|^2 (x - \lambda_j), \tag{6.7}$$

where $f(x)$ is the characteristic polynomial of A, $f_{(i)}(x)$ is the characteristic polynomial of $A(\{i\})$, $f_{[i]}(x)$ is the characteristic polynomial of $A[\{i\}]$, and U is any unitary matrix that diagonalizes A.

Since we are concerned here only with whether or not each principal submatrix is singular, we need only consider the constant terms of (6.6) and (6.7). The former is

$$\sum_{j=1}^{n} |U_{ij}|^2 (-1)^n (\det A)(-\lambda_j)^{-1} = (-1)^{n-1} (\det A) \sum_{j=1}^{n} |U_{ij}|^2 \frac{1}{\lambda_j}, \tag{6.8}$$

while the latter is

$$-\sum_{j=1}^{n} |U_{ij}|^2 \lambda_j. \tag{6.9}$$

In the special case where

$$\sum \lambda_i = \sum 1/\lambda_i = 0, \tag{6.10}$$

the values of (6.8) and (6.9) are both zero if every $|U_{ij}|$ is the same. (For this it is sufficient to take U as the $n \times n$ Fourier matrix defined by $U_{ij} = n^{-1/2}\,\omega^{(i-1)(j-1)}$, where ω is any primitive nth root of unity.) Thus, for $n = 5$, take U to be any such unitary matrix and take

$$\lambda_1 = -1, \quad \lambda_2 = -1, \quad \lambda_3 = -1, \quad \lambda_4 = \tfrac{3-\sqrt{5}}{2}, \quad \lambda_5 = 1/\lambda_4 = \tfrac{3+\sqrt{5}}{2}. \tag{6.11}$$

Then $\{\lambda_1, \ldots, \lambda_5\} = \{1/\lambda_1, \ldots, 1/\lambda_5\}$ and it is easy to see that (6.10) holds. Hence, if $D = (d_{ij})$ is the diagonal matrix defined by $d_{ii} = \lambda_i$ then $A = UDU^{-1}$ is a complex Hermitian matrix with principal rank characteristic sequence 101101. A similar argument, together with Lemmas 3.3 and 3.6, is enough to show that Question 6.6 has a negative answer in the complex Hermitian case for all $s \geq 2$.

To define $\mathrm{pr}(A)$ for a complex *symmetric* matrix A, it is possible to proceed with the straightforward generalization of Definition 1.1, or to generalize the definition of $\mathrm{pr}(A)$ via the alternative characterization given by (1.3). The following example from [12] shows that Theorem 6.3 will fail in either case.

Example 6.8 (Parker [12]). For any primitive third root of unity ω, let

$$A = \begin{bmatrix} 0 & 1 & 1 & \omega^2 & \omega \\ 1 & 0 & 1 & \omega & \omega^2 \\ 1 & 1 & 0 & 1 & 1 \\ \omega^2 & \omega & 1 & 0 & 1 \\ \omega & \omega^2 & 1 & 1 & 0 \end{bmatrix}.$$

It is easily checked that for each $k \in \{2, 3, 5\}$, this matrix A has both a nonsingular principal submatrix of order k and a principal submatrix of rank k. Meanwhile, A has neither a nonsingular principal submatrix of order 4 nor a principal submatrix of rank 4. Hence, $\mathrm{pr}(A) = 101101$ regardless of whether the definition of $\mathrm{pr}(A)$ is generalized via the original Definition 1.1 or via (1.3).

7. Sequences beginning with 1010

In this section we consider sequences beginning with 1010. Though it may seem highly restrictive to consider only sequences beginning with a given subsequence, results obtained about such sequences have the potential to provide information about the attainability of more general sequences. This potential is illustrated, for instance, by the proof of Theorem 4.6, wherein it is shown that the only way an attainable sequence containing σ as a subsequence may exist is if the reverse of σ occurs as the *initial* subsequence of some attainable sequence. A very similar technique underlies the proofs of this section. Note that Theorem 2.7 is essential to this sort of argument; it was applied in proving Theorem 4.6 and is applied in a similar way here to prove Theorem 7.2.

Lemma 7.1. *Let $k \geqslant 2$, and suppose the sequence $r_0r_1 \cdots r_k$ does not contain three consecutive 1s, has $r_k = 1$, and $r_0r_1r_2 \neq 110$. Then the sequence occurs as the initial subsequence of some attainable sequence if and only if it is itself attainable.*

Proof. One direction is trivial. For the other, suppose that for some $n \geqslant k$, there is some $n \times n$ real symmetric matrix A such that

$$\mathrm{pr}(A) = r_0r_1 \cdots r_kr_{k+1} \cdots r_n.$$

By Lemma 4.5, there is some principal submatrix A' of A with $\mathrm{pr}(A') = r_0'r_1r_2 \cdots r_k$ for some $r_0' \in \{0, 1\}$. If $r_0 = 0$, then $r_0' = 0 = r_0$. Hence, suppose $r_0 = 1$. By hypothesis, $r_0r_1r_2$ cannot be 110 or 111, nor can it be 100, as otherwise A' would violate Theorem 4.1. Thus, $r_0r_1r_2 = 101$, and so $r_1 = 0$, implying $r_0' = 1 = r_0$. $\square$

We may associate to any $n \times n$ real symmetric matrix A the unique undirected graph with vertex set $\{v_1, \ldots, v_n\}$ in which the vertices v_i and v_j are adjacent if and only if $a_{ij} = a_{ji}$ is nonzero. We refer to this as the *graph of A*. (Note that it may have loops.) Essential to the following proof is the observation that if $\mathrm{pr}(A)$ begins with 1010, then the graph of A must be loopless and triangle-free.

Theorem 7.2. *The sequence* 0110101 *does not occur as the initial subsequence of any attainable sequence.*

Proof. By Lemma 7.1, it suffices to show that 0110101 is itself not attainable. Thus, suppose for the sake of contradiction that there is some 6×6 real symmetric matrix B with $\mathrm{pr}(B) = 0110101$. Let $A = B^{-1}$. Then $\mathrm{pr}(A) = 1010111$, by Theorem 2.7, implying that the graph of A is loopless and triangle-free. Call this graph G. Note that G certainly cannot be combinatorially singular. Since B has no zero entry on its diagonal, every principal minor of A of order 5 must be nonzero, and hence no graph obtained by deleting a single vertex from G may be combinatorially singular either.

We proceed by examining the possibilities for the graph G. First, consider the size of a smallest connected component H of G. As G is not combinatorially singular, it may not have any isolated vertex, and hence H has at least two vertices. If H consists of a single edge, then deleting one of its vertices from G results in a graph that is combinatorially singular, so this is ruled out. If H has exactly 3 vertices, then, as G is triangle-free and has no isolated vertex, these vertices must be connected in a path of length two. But then deleting the middle vertex of this path from G results in a graph that is combinatorially singular. The only possibility remaining is that H has 6 vertices, meaning that G is connected.

Hence, G is a connected graph on 6 vertices that is triangle-free. There are only 5 such graphs. (For a list, see [14].) Moreover, G has no vertex of degree one, as deleting the neighbor of such a vertex would result in a graph with an isolated vertex, which would then be combinatorially singular. There must also be no vertex of G that occurs in every cycle, as the removal of such a vertex would result in a forest on an odd number of vertices, which necessarily is combinatorially singular.

There are only two possibilities remaining; either G is the complete bipartite graph $K_{3,3}$ or G is $K_{3,3}$ with a single edge deleted. If G is $K_{3,3}$, then deleting any single vertex from G results in the graph $K_{2,3}$, and it is easy to verify that this graph is combinatorially singular; as it has an odd number of vertices, any partition of its vertex set into cycles and edges must include at least one cycle, but that cycle must be a 4-cycle, leaving a single vertex that cannot be covered by an edge or cycle. On the other hand, if G is $K_{3,3}$ with a single edge removed, then removing one of the endpoints of that missing edge again results in $K_{2,3}$.

As we have now ruled out all possibilities for the graph of A, it follows that no such matrix B may exist. Hence, 0110101 is not attainable. $\square$

With Theorem 7.2 in hand, we can account for every sequence for $n = 6$, thereby determining PR_6; see Table 7.1.

Table 7.1
All attainable sequences for $n = 6$.

Sequence	Matrix	Result(s)
$r_0r_1r_2r_3r_4r_50$ with $r_0r_1r_2r_3r_4r_5$ attainable	-	2.6
0101011	$M_{0101011}$	*See below*
0101111	$J_6 - 2I_6$	2.2
0110111	$J_6 - 3I_6$	2.2
0111011	$J_6 - 4I_6$	2.2
0111101	$J_6 - 5I_6$	2.2
0111111	I_6	2.1(ii)
1010101	$A(P_6)$	3.3
1010111	$A(F_6)$	3.5
1011101	$A(V_6)$	3.6
1011111	$J_6 - I_6 = A(K_6)$	2.2
1101010	$(A(C_5))^{-1} \oplus O_1$	3.4, 2.7 and 2.3
1101110	$(J_5 - 2I_5) \oplus O_1$	2.2 and 2.3
1110101	$(A(F_6))^{-1}$	3.5
1110111	$Q_{6,1}$	3.7
1111011	$M_{1111011}$	*See below*
1111101	$Q_{6,2}$ or $(B_6)^{-1}$	3.7 or 2.8
1111111	$L_2 \oplus I_4$	2.5

$$M_{0101011} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 & 1 \end{bmatrix} \quad M_{1111011} = \begin{bmatrix} -3 & 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & -2 \\ 2 & 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

8. Sequences beginning with 010

In this section, we focus on sequences that begin with 010. These are of interest primarily because, as Proposition 8.1 shows, if such a sequence is attainable, it must be attainable by a matrix with a highly restricted form.

Proposition 8.1. *If a sequence beginning with* 010 *is attainable, then it is attainable by a matrix with every entry* ± 1 *and all entries in the first row, the first column, and the diagonal equal to* 1.

Proof. Suppose A is a matrix such that $\mathrm{pr}(A)$ has 010 as an initial subsequence. Let $D = [d_{ij}]$ be the $n \times n$ diagonal matrix defined by

$$d_{ii} = \mathrm{sign}(a_{1i})/\sqrt{|a_{ii}|}.$$

Let $B = [b_{ij}] = DAD$, and observe that every diagonal entry of B is equal to ± 1. Then, as multiplication of any row or column of a matrix by a nonzero constant preserves the rank of every submatrix, $\mathrm{pr}(B)$ also begins with 010. From this it follows that every diagonal entry of B must be the same, as if $b_{ii} = 1$ and $b_{jj} = -1$, then

$$B_{ij} = b_{ii}b_{jj} - b_{ij}b_{ji} = -1 - b_{ij}^2$$

cannot be zero. Thus, assume without loss of generality that every diagonal entry of B is 1. As every minor of B of order 2 is zero, it follows that each off-diagonal entry is ± 1. Then, by applying an appropriate signature similarity, each entry in the first row and column can be taken to be 1. $\square$

Table 8.1
All unattainable sequences beginning with 010 that are not accounted for by our results.

$n = 7$	$n = 9$	010 1111010	010 10111000	010 11101011
010 10111	010 1010110	010 1111011	010 10111010	010 11101111
$n = 8$	010 1010111	$n = 10$	010 10111011	010 11110100
010 101011	010 1011100	010 10101011	010 10111100	010 11110101
010 101110	010 1011101	010 10101100	010 10111101	010 11110110
010 101111	010 1011110	010 10101110	010 10111110	010 11110111
010 111101	010 1011111	010 10101111	010 10111111	010 11111101

For $n \leq 10$, the number of $n \times n$ matrices of the form given in the statement of Proposition 8.1 is small enough that they can be exhaustively checked by a computer in a reasonable amount of time. Performing this search resulted in a complete list (see Table 8.1) of sequences beginning with 010 that are both not attainable and not accounted for by any of the results of this paper.

9. Concluding remarks

In the characterization of the principal rank characteristic sequence given by (1.3), each 0 in the sequence indicates that every principal minor of the corresponding order is zero, while each 1 in the sequence indicates that some principal minor of the corresponding order is nonzero. If the presence of a 1 in the sequence were to indicate instead that *every* principal minor of the corresponding order is nonzero (keeping in mind that by convention the principal minor of order zero has the value 1), then the problem would be quite different. For example, it is easy to check that the sequence 1010 would no longer be attainable.

It may be interesting to study the inverse principal rank characteristic problem for restricted classes of matrices. Those matrices in which every entry is either 0 or 1 is one example, motivated by interest in graph eigenvalues. In our determination of PR_n for $n \leq 6$, we found that every sequence beginning with 10 is attainable by the adjacency matrix of some graph. It is an open question as to whether this holds for larger values of n.

Another class of interest is the set of matrices with entries from the set $\{1, -1\}$. This includes matrices of the form specified in Proposition 8.1, and one potential goal for future research would be to prove results that provide an explanation for why some of the sequences in Table 8.1 are not attainable.

A further open issue is to determine the precise set of values for which Question 6.6 has an affirmative answer. More generally, determine if it is possible to generalize Theorems 4.4 and 6.5 by finding a set $\{i_1, i_2, \ldots, i_s\}$ of size at least two such that for any $n \times n$ real symmetric matrix A with $\mathrm{pr}(A) = r_0 r_1 \cdots r_n$, if $r_{k+i_1} = r_{k+i_2} = \cdots r_{k+i_s} = 0$, for some k with $0 \leq k \leq n - i_s$, then $r_{k+i_s} = r_{k+i_s+1} = \cdots = r_n = 0$.

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